Fifth Edition

MODERN ENGINEERING MATHEMATICS

Glyn James

ALWAYS LEARNING

PEARSON

Modern Engineering Mathematics

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Fifth Edition

Glyn James *Coventry University*
and
David Burley *University of Sheffield*
Dick Clements *University of Bristol*
Phil Dyke *University of Plymouth*
John Searl *University of Edinburgh*
Jerry Wright *AT&T Shannon Laboratory*

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ON THE WEBSITE

For open-access **student resources** specifically written to complement this textbook and support your learning, please visit **www.pearsoned.co.uk/james**

Lecturer Resources

For password-protected online resources tailored to support the use of this textbook in teaching, please visit **www.pearsoned.co.uk/james**

Preface

As with the previous editions, the range of material covered in this fifth edition is regarded as appropriate for a first-level core studies course in mathematics for undergraduate courses in all engineering disciplines. Whilst designed primarily for use by engineering students it is believed that the book is also highly suitable for students of the physical sciences and applied mathematics. Additional material appropriate for second-level undergraduate core studies, or possibly elective studies for some engineering disciplines, is contained in the companion text *Advanced Modern Engineering Mathematics*.

The objective of the authoring team remains that of achieving a balance between the development of understanding and the mastering of solution techniques, with the emphasis being on the development of the student's ability to use mathematics with understanding to solve engineering problems. Consequently, the book is not a collection of recipes and techniques designed to teach students to solve routine exercises, nor is mathematical rigour introduced for its own sake. To achieve the desired objective the text contains:

- Worked examples
  Approximately 500 worked examples, many of which incorporate mathematical models and are designed both to provide relevance and to reinforce the role of mathematics in various branches of engineering. In response to feedback from users, additional worked examples have been incorporated within this revised edition.
- Applications
  To provide further exposure to the use of mathematical models in engineering practice, each chapter contains sections on engineering applications. These sections form an ideal framework for individual, or group, case study assignments leading to a written report and/or oral presentation, thereby helping to develop the skills of mathematical modelling necessary to prepare for the more open-ended modelling exercises at a later stage of the course.
- Exercises
  There are numerous exercise sections throughout the text, and at the end of each chapter there is a comprehensive set of review exercises. While many of the exercise problems are designed to develop skills in mathematical techniques, others are designed to develop understanding and to encourage learning by doing, and some are of an open-ended nature. This book contains over 1200 exercises and answers to all the questions are given. It is hoped that this provision, together with the large number of worked examples and style of presentation,

also makes the book suitable for private or directed study. Again in response to feedback from users, the frequency of exercises sections has been increased and additional questions have been added to many of the sections.

- Numerical methods
  Recognizing the increasing use of numerical methods in engineering practice, which often complement the use of analytical methods in analysis and design and are of ultimate relevance when solving complex engineering problems, there is wide agreement that they should be integrated within the mathematics curriculum. Consequently the treatment of numerical methods is integrated within the analytical work throughout the book.

Much of the feedback from users relates to the role and use of software packages, particularly symbolic algebra packages, in the teaching of mathematics to engineering students. In response, use of such packages continues to be a significant feature of this new edition. Whilst any appropriate software package can be used, the authors recommend the use of MATLAB and/or MAPLE and have continued to adopt their use in this text. Throughout, emphasis will be on the use of MATLAB, with reference made to corresponding MAPLE commands and differences in syntax highlighted. MATLAB/MAPLE commands have been introduced and illustrated, as inserts, throughout the text so that their use can be integrated into the teaching and learning processes. Students are strongly encouraged to use one of these packages to check the answers to the examples and exercises. It is stressed that the MATLAB/MAPLE inserts are not intended to be a first introduction of the package to students; it is anticipated that they will receive an introductory course elsewhere and will be made aware of the excellent 'help' facility available. The purpose of incorporating the inserts is not only to improve efficiency in the use of the package but also to provide a facility to help develop a better understanding of the related mathematics. Whilst use of such packages takes the tedium out of arithmetic and algebraic manipulations it is important that they are used to enhance understanding and not to avoid it. It is recognised that not all users of the text will have access to either MATLAB or MAPLE, and consequently all the inserts are highlighted and can be 'omitted' without loss of continuity in developing the subject content. Throughout the text two icons are used

- An open screen indicates that use of a software package would be useful (e.g. for checking solutions) but not essential
- A closed screen indicates that the use of a software package is essential or highly desirable.

Feedback, from users of the previous edition, on the subject content has been favourable, and consequently no new chapters have been introduced. However, in response to the feedback, chapters have been reviewed and amended/updated accordingly. Whilst subject content at this level has not changed much over the years the mode of delivery is being driven by developments in computer technology. Consequently there has been a shift towards online teaching and learning, coupled with student self-study programmes. In support of such programmes, worked examples and exercises sections are seen by many as the backbone of the text. Consequently in this new edition emphasis is given to strengthening the 'Worked Examples' throughout the text and increasing the frequency and number of questions in the 'Exercises Sections'. This has involved the restructuring, sometimes significant, of material within individual chapters.

A comprehensive Solutions Manual is obtainable free of charge to lecturers using this textbook. It will also be available for download via the Web at www.pearsoned.co.uk/james.

Also available online is a set of 'Refresher Units' covering topics students should have encountered at school but may not have used for some time.

Acknowledgements

The authoring team is extremely grateful to all the reviewers and users of the text who have provided valuable comments on previous editions of this book. Most of this has been highly constructive and very much appreciated. The team has continued to enjoy the full support of a very enthusiastic production team at Pearson Education and wishes to thank all those concerned. Finally I would like to thank my wife, Dolan, for her full support throughout the preparation of this text and its previous editions.

Glyn James
Coventry
February 2015

About the Authors

Glyn James retired as Dean of the School of Mathematical and Information Sciences at Coventry University in 2001 and is now Emeritus Professor in Mathematics at the University. He graduated from the University College of Wales, Cardiff in the late 1950s, obtaining first class honours degrees in both Mathematics and Chemistry. He obtained a PhD in Engineering Science in 1971 as an external student of the University of Warwick. He has been employed at Coventry since 1964 and held the position of Head of the Mathematics Department prior to his appointment as Dean in 1992. His research interests are in control theory and its applications to industrial problems. He also has a keen interest in mathematical education, particularly in relation to the teaching of engineering mathematics and mathematical modelling. He was co-chairman of the European Mathematics Working Group established by the European Society for Engineering Education (SEFI) in 1982, a past chairman of the Education Committee of the Institute of Mathematics and its Applications (IMA), and a member of the Royal Society Mathematics Education Subcommittee. In 1995 he was chairman of the Working Group that produced the report ‘Mathematics Matters in Engineering’ on behalf of the professional bodies in engineering and mathematics within the UK. He is also a member of the editorial/advisory board of three international journals. He has published numerous papers and is co-editor of five books on various aspects of mathematical modelling. He is a past Vice-President of the IMA and has also served a period as Honorary Secretary of the Institute. He is a Chartered Mathematician and a Fellow of the IMA.

David Burley retired from the University of Sheffield in 1998. He graduated in mathematics from King’s College, University of London in 1955 and obtained his PhD in mathematical physics. After working in the University of Glasgow, he spent most of his academic career in the University of Sheffield, being Head of Department for six years. He has long experience of teaching engineering students and has been particularly interested in encouraging students to construct mathematical models in physical and biological contexts to enhance their learning. His research work has ranged through statistical mechanics, optimization and fluid mechanics. He has particular interest in the flow of molten glass in a variety of situations and the application of results in the glass industry. Currently he is involved in a large project concerning heat transfer problems in the deep burial of nuclear waste.

Dick Clements is an Emeritus Professor at the University of Bristol, having previously lectured in the Department of Engineering Mathematics (1973–2007). He has an MA in Mathematics and a PhD in Aeronautical Engineering from the University of Cambridge. He has undertaken research in a wide range of engineering topics but is particularly interested in mathematical modelling and in new approaches to the teaching of mathematics to engineering students. He has published numerous papers and one previous book, *Mathematical Modelling: A Case Study Approach*. He is a Chartered Engineer, a Chartered Mathematician, a member of the Royal Aeronautical Society, a Fellow of the Institute of Mathematics and Its Applications, an Associate Fellow of the Royal Institute of Navigation, and a Fellow of the Higher Education Academy.

Phil Dyke is Professor of Applied Mathematics and Head of the School of Mathematics and Statistics at the University of Plymouth. After graduating with first class honours in Mathematics from the University of London, he gained a PhD in coastal sea modelling at Reading in 1972. Since then, Phil Dyke has been a full-time academic, initially at Heriot-Watt University teaching engineers followed by a brief spell at Sunderland. He has been at Plymouth since 1984. He still engages in teaching and is actively involved in building mathematical models relevant to environmental issues.

John Searl was Director of the Edinburgh Centre for Mathematical Education at the University of Edinburgh before his retirement. As well as lecturing on mathematical education, he taught service courses for engineers and scientists. His most recent research concerned the development of learning environments that make for the effective learning of mathematics for 16–20-year-olds. As an applied mathematician who worked collaboratively with (among others) engineers, physicists, biologists and pharmacologists, he is keen to develop the problem-solving skills of students and to provide them with opportunities to display their mathematical knowledge within a variety of practical contexts. These contexts develop the extended reasoning needed in all fields of engineering.

Jerry Wright was a Lead Member of Technical Staff at the AT&T Shannon Laboratory, New Jersey, USA until he retired in 2012. He graduated in Engineering (BSc and PhD at the University of Southampton) and in Mathematics (MSc at the University of London) and worked at the National Physical Laboratory before moving to the University of Bristol in 1978. There he acquired wide experience in the teaching of mathematics to students of engineering, and became Senior Lecturer in Engineering Mathematics. He held a Royal Society Industrial Fellowship for 1994, and is a Fellow of the Institute of Mathematics and its Applications. In 1996 he moved to AT&T Labs (formerly part of Bell labs) to continue his research in spoken language understanding, human/computer dialogue systems, and data mining.

1 Number, Algebra and Geometry

Chapter 1 Contents

1.1 Introduction

Mathematics plays an important role in our lives. It is used in everyday activities from buying food to organizing maintenance schedules for aircraft. Through applications developed in various cultural and historical contexts, mathematics has been one of the decisive factors in shaping the modern world. It continues to grow and to find new uses, particularly in engineering and technology.

Mathematics provides a powerful, concise and unambiguous way of organizing and communicating information. It is a means by which aspects of the physical universe can be explained and predicted. It is a problem-solving activity supported by a body of knowledge. Mathematics consists of facts, concepts, skills and thinking processes – aspects that are closely interrelated. It is a hierarchical subject in that new ideas and skills are developed from existing ones. This sometimes makes it a difficult subject for learners who, at every stage of their mathematical development, need to have ready recall of material learned earlier.

In the first two chapters we shall summarize the concepts and techniques that most students will already understand and we shall extend them into further developments in mathematics. There are four key areas of which students will already have considerable knowledge.

- numbers
- algebra
- geometry
- functions

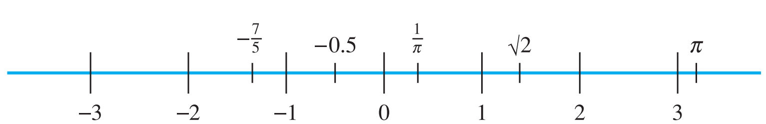
These areas are vital to making progress in engineering mathematics (indeed, they will solve many important problems in engineering). Here we will aim to consolidate that knowledge, to make it more precise and to develop it. In this first chapter we will deal with the first three topics; functions are considered in Chapter 2.

1.2 Number and arithmetic

1.2.1 Number line

Mathematics has grown from primitive arithmetic and geometry into a vast body of knowledge. The most ancient mathematical skill is counting, using, in the first instance, the natural numbers and later the integers. The term **natural numbers** commonly refers to the set $\mathbb{N} = \{1, 2, 3, \ldots\}$, and the term **integers** to the set $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$. The integers can be represented as equally spaced points on a line called the **number line** as shown in Figure 1.1. In a computer the integers can be stored exactly. The set of all points (not just those representing integers) on the number line represents the **real numbers** (so named to distinguish them from the complex numbers, which are

Figure 1.1
The number line.



discussed in Chapter 3). The set of real numbers is denoted by $\mathbb{R}$. The general real number is usually denoted by the letter x and we write 'x in $\mathbb{R}$', meaning x is a real number. A real number that can be written as the ratio of two integers, like $\frac{3}{2}$ or $-\frac{7}{5}$, is called a **rational number**. Other numbers, like $\sqrt{2}$ and π, that cannot be expressed in that way are called **irrational numbers**. In a computer the real numbers can be stored only to a limited number of figures. This is a basic difference between the ways in which computers treat integers and real numbers, and is the reason why the computer languages commonly used by engineers distinguish between integer values and variables on the one hand and real number values and variables on the other.

1.2.2 Representation of numbers

For everyday purposes we use a system of representation based on ten **numerals**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. These ten symbols are sufficient to represent all numbers if a **position notation** is adopted. For whole numbers this means that, starting from the right-hand end of the number, the least significant end, the figures represent the number of units, tens, hundreds, thousands, and so on. Thus one thousand, three hundred and sixty-five is represented by 1365, and two hundred and nine is represented by 209. Notice the role of the 0 in the latter example, acting as a position keeper. The use of a decimal point makes it possible to represent fractions as well as whole numbers. This system uses ten symbols. The number system is said to be 'to base ten' and is called the **decimal** system. Other bases are possible: for example, the Babylonians used a number system to base sixty, a fact that still influences our measurement of time. In some societies a number system evolved with more than one base, a survival of which can be seen in imperial measures (inches, feet, yards, …). For some applications it is more convenient to use a base other than ten. Early electronic computers used **binary** numbers (to base two); modern computers use **hexadecimal** numbers (to base sixteen). For elementary (pen-and-paper) arithmetic a representation to base twelve would be more convenient than the usual decimal notation because twelve has more integer divisors (2, 3, 4, 6) than ten (2, 5).

In a decimal number the positions to the left of the decimal point represent units (10^0), tens (10^1), hundreds (10^2) and so on, while those to the right of the decimal point represent tenths (10^{-1}), hundredths (10^{-2}) and so on. Thus, for example

$$\begin{array}{ccccccc} 2 & 1 & 4 & \cdot & 3 & 6 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ 10^2 & 10^1 & 10^0 & & 10^{-1} & 10^{-2} \end{array}$$

so

$$\begin{aligned} 214.36 &= 2(10^2) + 1(10^1) + 4(10^0) + 3(\tfrac{1}{10}) + 6(\tfrac{1}{100}) \\ &= 200 + 10 + 4 + \tfrac{3}{10} + \tfrac{6}{100} \\ &= \tfrac{21436}{100} = \tfrac{5359}{25} \end{aligned}$$

In other number bases the pattern is the same: in base b the position values are b^0, b^1, b^2, … and b^{-1}, b^{-2}, … . Thus in binary (base two) the position values are units, twos, fours, eights, sixteens and so on, and halves, quarters, eighths and so on. In hexadecimal (base sixteen) the position values are units, sixteens, two hundred and fifty-sixes and so on, and sixteenths, two hundred and fifty-sixths and so on.

Example 1.1

Write (a) the binary number 1011101_2 as a decimal number and (b) the decimal number 115_{10} as a binary number.

Solution

(a) $1011101_2 = 1(2^6) + 0(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$

$= 64_{10} + 0 + 16_{10} + 8_{10} + 4_{10} + 0 + 1_{10}$

$= 93_{10}$

(b) We achieve the conversion to binary by repeated division by 2. Thus

$115 \div 2 = 57$	remainder 1	(2^0)
$57 \div 2 = 28$	remainder 1	(2^1)
$28 \div 2 = 14$	remainder 0	(2^2)
$14 \div 2 = 7$	remainder 0	(2^3)
$7 \div 2 = 3$	remainder 1	(2^4)
$3 \div 2 = 1$	remainder 1	(2^5)
$1 \div 2 = 0$	remainder 1	(2^6)

so that

$115_{10} = 1110011_2$

Example 1.2

Represent the numbers (a) two hundred and one, (b) two hundred and seventy-five, (c) five and three-quarters and (d) one-third in

(i) decimal form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;

(ii) binary form using the figures 0, 1;

(iii) duodecimal (base 12) form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, Δ, ε.

Solution

(a) two hundred and one

(i) = 2 (hundreds) + 0 (tens) and 1 (units) = 201_{10}

(ii) = 1 (one hundred and twenty-eight) + 1 (sixty-four) + 1 (eight) + 1 (unit) = 11001001_2

(iii) = 1 (gross) + 4 (dozens) + 9 (units) = 149_{12}

Here the subscripts 10, 2, 12 indicate the number base.

(b) two hundred and seventy-five

(i) = 2 (hundreds) + 7 (tens) + 5 (units) = 275_{10}

(ii) = 1 (two hundred and fifty-six) + 1 (sixteen) + 1 (two) + 1 (unit) = 100010011_2

(iii) $= 1$ (gross) $+ 10$ (dozens) $+$ eleven (units) $= 1\Delta\varepsilon_{12}$
(Δ represents ten and ε represents eleven)

(c) five and three-quarters

(i) $= 5$ (units) $+ 7$ (tenths) $+ 5$ (hundredths) $= 5.75_{10}$

(ii) $= 1$ (four) $+ 1$ (unit) $+ 1$ (half) $+ 1$ (quarter) $= 101.11_2$

(iii) $= 5$ (units) $+ 9$ (twelfths) $= 5.9_{12}$

(d) one-third

(i) $= 3$ (tenths) $+ 3$ (hundredths) $+ 3$ (thousandths) $+ \ldots = 0.333 \ldots_{10}$

(ii) $= 1$ (quarter) $+ 1$ (sixteenth) $+ 1$ (sixty-fourth) $+ \ldots = 0.010101 \ldots_2$

(iii) $= 4$ (twelfths) $= 0.4_{12}$

1.2.3 Rules of arithmetic

The basic arithmetical operations of addition, subtraction, multiplication and division are performed subject to the **Fundamental Rules of Arithmetic**. For any three numbers a, b and c:

(a1) the commutative law of addition

$$a + b = b + a$$

(a2) the commutative law of multiplication

$$a \times b = b \times a$$

(b1) the associative law of addition

$$(a + b) + c = a + (b + c)$$

(b2) the associative law of multiplication

$$(a \times b) \times c = a \times (b \times c)$$

(c1) the distributive law of multiplication over addition and subtraction

$$(a + b) \times c = (a \times c) + (b \times c)$$

$$(a - b) \times c = (a \times c) - (b \times c)$$

(c2) the distributive law of division over addition and subtraction

$$(a + b) \div c = (a \div c) + (b \div c)$$

$$(a - b) \div c = (a \div c) - (b \div c)$$

Here the brackets indicate which operation is performed first. These operations are called **binary** operations because they associate with every two members of the set of real numbers a unique third member; for example,

$$2 + 5 = 7 \quad \text{and} \quad 3 \times 6 = 18$$

Example 1.3 Find the value of $(100 + 20 + 3) \times 456$.

Solution Using the distributive law we have

$$\begin{aligned}(100 + 20 + 3) \times 456 &= 100 \times 456 + 20 \times 456 + 3 \times 456 \\ &= 45\,600 + 9120 + 1368 = 56\,088\end{aligned}$$

Here 100×456 has been evaluated as

$$100 \times 400 + 100 \times 50 + 100 \times 6$$

and similarly 20×456 and 3×456.

This, of course, is normally set out in the traditional school arithmetic way:

$$\begin{array}{r} 456 \\ \underline{123\,\times} \\ 1\,368 \\ 9\,120 \\ \underline{45\,600} \\ \underline{56\,088} \end{array}$$

Example 1.4 Rewrite $(a + b) \times (c + d)$ as the sum of products.

Solution Using the distributive law we have

$$\begin{aligned}(a + b) \times (c + d) &= a \times (c + d) + b \times (c + d) \\ &= (c + d) \times a + (c + d) \times b \\ &= c \times a + d \times a + c \times b + d \times b \\ &= a \times c + a \times d + b \times c + b \times d\end{aligned}$$

applying the commutative laws several times.

A further operation used with real numbers is that of **powering**. For example, $a \times a$ is written as a^2, and $a \times a \times a$ is written as a^3. In general the product of n a's where n is a positive integer is written as a^n. (Here the n is called the **index** or **exponent**.) Operations with powering also obey simple rules:

$$a^n \times a^m = a^{n+m} \tag{1.1a}$$

$$a^n \div a^m = a^{n-m} \tag{1.1b}$$

$$(a^n)^m = a^{nm} \tag{1.1c}$$

From rule (1.1b) it follows, by setting $n = m$ and $a \neq 0$, that $a^0 = 1$. It is also convention to take $0^0 = 1$. The process of powering can be extended to include the fractional powers like $a^{1/2}$. Using rule (1.1c),

$$(a^{1/n})^n = a^{n/n} = a^1$$

and we see that

$$a^{1/n} = \sqrt[n]{a}$$

the nth root of a. Also, we can define a^{-m} using rule (1.1b) with $n = 0$, giving

$$1 \div a^m = a^{-m}, \qquad a \neq 0$$

Thus a^{-m} is the reciprocal of a^m. In contrast with the binary operations +, ×, − and ÷, which operate on two numbers, the powering operation $(\ )^r$ operates on just one element and is consequently called a **unary** operation. Notice that the fractional power

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{(a^m)}$$

is the nth root of a^m. If n is an even integer, then $a^{m/n}$ is not defined when a is negative. When $\sqrt[n]{a}$ is an irrational number then such a root is called a **surd**.

Numbers like $\sqrt{2}$ were described by the Greeks as **a-logos**, without a ratio number. An Arabic translator took the alternative meaning 'without a word' and used the arabic word for 'deaf', which subsequently became **surdus**, Latin for deaf, when translated from Arabic to Latin in the mid-twelfth century.

Example 1.5

Find the values of

(a) $27^{1/3}$ (b) $(-8)^{2/3}$ (c) $16^{-3/2}$

(d) $(-2)^{-2}$ (e) $(-1/8)^{-2/3}$ (f) $(9)^{-1/2}$

Solution

(a) $27^{1/3} = \sqrt[3]{27} = 3$

(b) $(-8)^{2/3} = (\sqrt[3]{(-8)})^2 = (-2)^2 = 4$

(c) $16^{-3/2} = (16^{1/2})^{-3} = (4)^{-3} = \frac{1}{4^3} = \frac{1}{64}$

(d) $(-2)^{-2} = \dfrac{1}{(-2)^2} = \frac{1}{4}$

(e) $(-1/8)^{-2/3} = [\sqrt[3]{(-1/8)}]^{-2} = [\sqrt[3]{(-1)}/\sqrt[3]{(8)}]^{-2} = [-1/2]^{-2} = 4$

(f) $(9)^{-1/2} = (3)^{-1} = \frac{1}{3}$

Example 1.6

Express (a) in terms of $\sqrt{2}$ and simplify (b) to (f).

(a) $\sqrt{18} + \sqrt{32} - \sqrt{50}$ (b) $6/\sqrt{2}$ (c) $(1 - \sqrt{3})(1 + \sqrt{3})$

(d) $\dfrac{2}{1 - \sqrt{3}}$ (e) $(1 + \sqrt{6})(1 - \sqrt{6})$ (f) $\dfrac{1 - \sqrt{2}}{1 + \sqrt{6}}$

Solution (a) $\sqrt{18} = \sqrt{(2 \times 9)} = \sqrt{2} \times \sqrt{9} = 3\sqrt{2}$

$\sqrt{32} = \sqrt{(2 \times 16)} = \sqrt{2} \times \sqrt{16} = 4\sqrt{2}$

$\sqrt{50} = \sqrt{(2 \times 25)} = \sqrt{2} \times \sqrt{25} = 5\sqrt{2}$

Thus $\sqrt{18} + \sqrt{32} - \sqrt{50} = 2\sqrt{2}$.

(b) $6/\sqrt{2} = 3 \times 2/\sqrt{2}$

Since $2 = \sqrt{2} \times \sqrt{2}$, we have $6/\sqrt{2} = 3\sqrt{2}$.

(c) $(1 - \sqrt{3})(1 + \sqrt{3}) = 1 + \sqrt{3} - \sqrt{3} - 3 = -2$

(d) Using the result of part (c) $\dfrac{2}{1 - \sqrt{3}}$ can be simplified by multiplying 'top and bottom' by $1 + \sqrt{3}$ (notice the sign change in front of the $\sqrt{}$). Thus

$$\begin{aligned}\frac{2}{1 - \sqrt{3}} &= \frac{2(1 + \sqrt{3})}{(1 - \sqrt{3})(1 + \sqrt{3})} \\ &= \frac{2(1 + \sqrt{3})}{1 - 3} \\ &= -1 - \sqrt{3}\end{aligned}$$

(e) $(1 + \sqrt{6})(1 - \sqrt{6}) = 1 - \sqrt{6} + \sqrt{6} - 6 = -5$

(f) Using the same technique as in part (d) we have

$$\begin{aligned}\frac{1 - \sqrt{2}}{1 + \sqrt{6}} &= \frac{(1 - \sqrt{2})(1 - \sqrt{6})}{(1 + \sqrt{6})(1 - \sqrt{6})} \\ &= \frac{1 - \sqrt{2} - \sqrt{6} + \sqrt{12}}{1 - 6} \\ &= -(1 - \sqrt{2} - \sqrt{6} + 2\sqrt{3})/5\end{aligned}$$

This process of expressing the irrational number so that all of the surds are in the numerator is called **rationalization**.

When evaluating arithmetical expressions the following rules of precedence are observed:

- the powering operation $(\ )^r$ is performed first
- then multiplication $\times$ and/or division $\div$
- then addition $+$ and/or subtraction $-$

When two operators of equal precedence are adjacent in an expression the left-hand operation is performed first. For example

$$12 - 4 + 13 = 8 + 13 = 21$$

and

$$15 \div 3 \times 2 = 5 \times 2 = 10$$

The precedence rules are overridden by brackets; thus

$$12 - (4 + 13) = 12 - 17 = -5$$

and

$$15 \div (3 \times 2) = 15 \div 6 = 2.5$$

Example 1.7 Evaluate $7 - 5 \times 3 \div 2^2$.

Solution Following the rules of precedence, we have

$$7 - 5 \times 3 \div 2^2 = 7 - 5 \times 3 \div 4 = 7 - 15 \div 4 = 7 - 3.75 = 3.25$$

1.2.4 Exercises

1. Find the decimal equivalent of 110110.101_2.

2. Find the binary and octal (base eight) equivalents of the decimal number 16 321. Obtain a simple rule that relates these two representations of the number, and hence write down the octal equivalent of 1011100101101_2.

3. Find the binary and octal equivalents of the decimal number 30.6. Does the rule obtained in Question 2 still apply?

4. Use binary arithmetic to evaluate

   (a) $100011.011_2 + 1011.001_2$

   (b) $111.10011_2 \times 10.111_2$

5. Simplify the following expressions, giving the answers with positive indices and without brackets:

   (a) $2^3 \times 2^{-4}$ (b) $2^3 \div 2^{-4}$ (c) $(2^3)^{-4}$

   (d) $3^{1/3} \times 3^{5/3}$ (e) $(36)^{-1/2}$ (f) $16^{3/4}$

6. The expression $7 - 2 \times 3^2 + 8$ may be evaluated using the usual implicit rules of precedence. It could be rewritten as $((7 - (2 \times (3^2))) + 8)$ using brackets to make the precedence explicit. Similarly rewrite the following expressions in fully bracketed form:

   (a) $21 + 4 \times 3 \div 2$

   (b) $17 - 6^{2^{+3}}$

   (c) $4 \times 2^3 - 7 \div 6 \times 2$

   (d) $2 \times 3 - 6 \div 4 + 3^{2^{-5}}$

7. Express the following in the form $x + y\sqrt{2}$ with x and y rational numbers:

   (a) $(7 + 5\sqrt{2})^3$ (b) $(2 + \sqrt{2})^4$

   (c) $\sqrt[3]{(7 + 5\sqrt{2})}$ (d) $\sqrt{(\frac{11}{2} - 3\sqrt{2})}$

8. Show that

   $$\frac{1}{a + b\sqrt{c}} = \frac{a - b\sqrt{c}}{a^2 - b^2c}$$

   Hence express the following numbers in the form $x + y\sqrt{n}$ where x and y are rational numbers and n is an integer:

   (a) $\dfrac{1}{7 + 5\sqrt{2}}$ (b) $\dfrac{2 + 3\sqrt{2}}{9 - 7\sqrt{2}}$

   (c) $\dfrac{4 - 2\sqrt{3}}{7 - 3\sqrt{3}}$ (d) $\dfrac{2 + 4\sqrt{5}}{4 - \sqrt{5}}$

9. Find the difference between 2 and the squares of

   $$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

   (a) Verify that successive terms of the sequence stand in relation to each other as m/n does to $(m + 2n)/(m + n)$.

   (b) Verify that if m/n is a good approximation to $\sqrt{2}$ then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions.

   (c) Find the next three terms of the above sequence.